FASnet21

the Workshop on
Formal and Analytic Solutions of Diff. Equations
on the Internet

June 28-July 2, 2021

UAH, Spain

Universidad de Alcalá

Contents

About

FASnet21

The Workshop FASnet21 (Formal and Analytic Solutions of Diff. (differential, partial differential, difference, q-difference, q-differencedifferential, ...) Equations on the Internet), will be an online meeting from June 28th to July 2nd, 2021. It aims to bring together experts in the field of formal and analytic solutions of functional equations, such as differential, partial differential, difference, q-difference, q-difference-differential equations.

This meeting expects to be an opportunity for the exchange of recent results in the field and to exhibit different possible directions of future research. One of its main objectives is to promote both new and existing scientific collaborations of the researchers in these topics.

Topics

- Ordinary differential equations in the complex domain. Formal and analytic solutions. Stokes multipliers.
- Formal and analytic solutions of partial differential equations.
- Formal and analytic solutions of difference equations (including q-difference and differential-difference equations). Special functions (hypergeometric functions and others), orthogonal polynomials, continuous and discrete Painleve equations.
- Integrable systems.
- Holomorphic vector fields. Normal forms.
- Asymptotic expansions, Borel summability.

Scientific Committee

Galina Filipuk	(University of Warsaw, Poland)
Alberto Lastra	(University of Alcalá, Spain)
Stephane Malek	(University of Lille, France)
Jorge Mozo	(University of Valladolid, Spain)
Yasunori Okada	(Chiba University, Japan)
Hidetoshi Tahara	(Sophia University, Japan)

Organizing Committee

Javier Jiménez-Garrido	(University of Cantabria, Spain)	jimenezjj@unican.es
Alberto Lastra	(University of Alcalá, Spain)	alberto.lastra@uah.es
Ignacio Miguel	(University of Valladolid, Spain)	ignacio.miguel@uva.es
Andrew Pickering	(Universidad Rey Juan Carlos, Spain)	andrew.pickering@urjc.es

Proceedings

We are pleased to announce that an agreement with the **American Mathematical Society** has been approved in order to publish a volume of proceedings of the conference FASnet21, in Contemporary Mathematics series.

Any interested participant is encouraged to submit a work for the volume of proceedings.

In order to estimate the number of contributions, we ask the interested participants to send an email to alberto.lastra@uah.es with some tentative information of the work, if possible (tentative authors, title, tentative number of pages).

The volume will consist of accepted works after a peer-review process.

The deadline for paper submission is **November 30, 2021**.

Templates and other information on this concern will be provided to interested participants. Please, do not hesitate to contact us.

List of Abstracts

Invariant Hermitian form of monodromy of spectral type $(111, 111, 111)$

Shunya Adachi, Kumamoto University, Kumamoto, Japan

We consider a Fuchsian differential equation on $\mathbb{P}^1$ of rank three with three regular singular points of spectral type $(111, 111, 111)$.

$$\frac{du}{dx} = \left(\frac{\tilde{A}}{x - t_1} + \frac{\tilde{B}}{x - t_2}\right) u, \quad \tilde{A}, \tilde{B} \in \mathrm{Mat}(3, \mathbb{C}), \tag{1}$$

This equation is non-rigid Fuchsian, indeed, two accessory parameters appear in (1) and the dimension of moduli space of the equations (1) is two.

The goal of this talk is to show that, there exists a class in the moduli space of the equations (1) such that they have monodromy invariant Hermitian forms.

In our knowledge, for the differential equations of the spectral type $(111, 111, 111)$, the existence of the monodromy invariant Hermitian form is known only for the Dotsenko-Fateev equation [Nuclear Phys. B **240**, 1984], which is a special case such that it has an integral representation of solutions of Euler type.

We show the existence of the monodromy invariant Hermitians form for a one parameter family of differential equations of spectral type $(111, 111, 111)$ without assuming the existence of integral representation of solutions. Then our result may determine a new class in non-rigid differential equations. This talk is based on a joint work with Yoshishige Haraoka (Kumamoto University, Japan).

Computation of homological equation for Hamiltonian normal form

A. B. Batkhin, Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia

We consider an algorithm for constructing and solving a homological equation of arbitrary order arising in the problem of reducing a Hamilton system to its normal form (NF) in the vicinity of a stationary points (SP) or of a periodic solution. Hamiltonian H near the SP can be written in the form

$$H(\mathbf{x}, \mathbf{y}) = H_0 + F = H_0 + \sum_{j=1}^{\infty} \varepsilon^j H_j(\mathbf{x}, \mathbf{y}),$$

where H_j is a homogeneous form of order $j+2$: $H_j = \sum\limits_{|\mathbf{p}|+|\mathbf{q}|=j+2} H_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}$.

To reduced the Hamiltonian $H(\mathbf{x}, \mathbf{y})$ to its complex NF [1] in the case, when all eigenvalues λ_j, $j = 1, \ldots, n$ of the form H_0 are pure imagine and simple,

$$h(\mathbf{z}, \bar{\mathbf{z}}) = h_0 + f = h_0 + \sum_{j=1}^{\infty} h_j(\mathbf{z}, \bar{\mathbf{z}}), \tag{1}$$

where $h_0 = \sum_{j=1} \lambda_j z_j \bar{z}_j$ and forms h_j, $j > 0$, contain only resonant terms $h_{\mathbf{pq}} \mathbf{z}^{\mathbf{p}} \bar{\mathbf{z}}^{\mathbf{q}}$, $|\mathbf{p}| + |\mathbf{q}| = j + 2$, such that $\sum_{j=1}^{n} \lambda_j (p_j - q_j) = 0$, one has to solve so called *homological equation*

$$f = h_0 * G + M, \quad M = F + F * G + \sum_{j=2} \frac{1}{j!} H * G^j, \tag{2}$$

where G is Lie generator $G = \sum_{j=1} \varepsilon^j G_j$ and $H * G^k \equiv H * G * G^{k-1}$ is k-th iteration of the Poisson bracket. Equation (2) can be written as the recurrent system of equation $h_j = h_0 * G_j + M_j$, $j = 1, 2, \ldots$, where the term M_j depends on F_k, f_k, G_k, $k < j$, obtained on the previous steps. The number of term in M_j can be reduced up to four times using substitutions $h_0 * G_k = f_k - M_k$, $k < j$.

Statement 1. *Let denote by* $f_k^+ \stackrel{\text{def}}{=} F_k + f_k$, $f_k^- = F_k - f_k$ *and* $H * G^k_{j_1 \cdots j_k} = H * G^{k-1}_{j_1 \cdots j_{k-1}} * G_{j_k}$, *then for each* $j > 2$ *term* M_j *of the homological equation* (2) *can be written in the form*

$$M_j = F_j + \frac{1}{2} \sum_{k=1}^{j-1} f_k^+ * G_{j-k} + \sum_{k=1}^{[j/2]} \alpha_{2k+1} \sum_{(i_1, \ldots, i_{2k+1}) \in \mu^j_{2k+1}} f^-_{i_1} * G^{2k}_{i_2 \cdots i_{2k+1}},$$

where μ^j_{2k+1} *is the set of all permutations of partitions* $\lambda_{2k+1}(j)$ *of integer number* j *into exactly* $2k+1$ *terms, i.e.* $j_1 + \cdots + j_{2k+1} = j$, *and* α_k *are coefficients, obtained by the generating function* $\mathfrak{g}(\varepsilon) = \varepsilon^2 \left(\frac{1}{2} - \frac{1}{\varepsilon} + \frac{1}{\mathrm{e}^{\varepsilon} - 1} \right)$. *For* $j = 1, 2$ *we have* $M_1 = F_1$ *and* $M_2 = F_2 + \frac{1}{2} f_1^+ * G_1$.

Each homological equation can be effectively solved algebraically by the method of invariant normalization [2]. Obtained up to the certain order NF (1) can be used in studying the stability of the corresponding invariant manifold, local integration of the system, searching for periodic solutions and first integrals and for asymptotic integration.

References

[1] Bruno, A. D. (1994). *The Restricted 3–body Problem: Plane Periodic Orbits*, Walter de Gruyter, Berlin.
[2] Zhuravlev, V. F., Petrov, A. G. and Shunderyuk, M. M. (2015). *Selected Problems of Hamiltonian Mechanics*, LENAND, Moscow, (in Russian).

Intermediate Heisenberg ferromagnet equations

B. Berntson, Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden

We present nonlocal integrable generalizations of the Heisenberg ferro- magnet equation. The new equations interpolate between Heisenberg ferromagnet equation and the half-wave maps equation, a recently-introduced nonlocal integrable equation related to the trigonometric spin Calogero-Moser system. Our main result is the identification of a new equation, called the non-chiral intermediate Heisenberg ferromagnet equation, which is similarly related to the elliptic spin Calogero-Moser system.

Generalized normal form of ODE systems

A. D. Bruno, Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia

Let in $\mathbb{R}^n$ be: the lattice L and the convex cone C with the vertex in the origin. We consider the formal ODE system

$$\dot{x}_i = x_i f_i(X) \stackrel{\text{def}}{=} x_i \sum f_{iQ} X^Q, \quad i = 1, \ldots, n, \tag{0.1}$$

where $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $Q = (q_1, \ldots, q_n) \in L \cap C$, $X^Q = x_1^{q_1} \cdots x_n^{q_n}$, constant coefficients $f_{iQ} \in \mathbb{C}$, $(f_{1O}, \ldots, f_{nO}) \stackrel{\text{def}}{=} \Lambda$.

Theorem 1. *There exists the formal invertible change of coordinates*

$$x_i = y_i g_i(Y) \stackrel{\text{def}}{=} y_i \sum g_{iQ} Y^Q, \quad i = 1, \ldots, n, \tag{0.2}$$

where $Q \in L \cap C$, $g_{iQ} \in \mathbb{C}$, *all* $g_{iO} = 1$, *which transforms the system* (0.1) *into generalized normal form*

$$\dot{y}_i = y_i h_i(Y) \stackrel{\text{def}}{=} y_i \sum h_{iQ} Y^Q, \quad i = 1, \ldots, n, \tag{0.3}$$

where $Q \in L \cap C$ *and presents only resonant terms* $h_{iQ} Y^Q$ *with*

$$\langle Q, \Lambda \rangle \stackrel{\text{def}}{=} \sum_{i=1}^{n} q_i \lambda_i = 0.$$

If the cone C is given by inequalities

$$\langle Q, K_j \rangle \geqslant 0, \quad j = 1, \ldots, m,$$

then the systems (0.1), (0.3) and the change (0.2) have a sense in domains

$$\mathcal{U}(\varepsilon, X) \stackrel{\text{def}}{=} \left\{ X : |X|^{K_j} \leqslant \varepsilon, \ j = 1, \ldots, m \right\}$$

and $\mathcal{U}(\varepsilon, Y)$, where ε is a small number.
For the case when $L = \mathbb{Z}^n$ and all $x_i f_i(X)$ are usual power series without free terms, a similar theorem was firstly stated and proved in [1]. In [2, Part I, Ch. II, § 2] it was given for $n = 2$, $L = \mathbb{Z}^2$ and arbitrary cone C. There it was used for a study of a structure of solutions of an ODE system near its very degenerate stationary point.

References

[1] Bruno A.D. Normal form of differential equations // Soviet Math. Dokl. 5 (1964) 1105–1108
[2] Bruno A.D. Local Methods in Nonlinear Differential Equations. Springer–Verlag: Berlin–Heidelberg–New York–London–Paris–Tokyo, 1989. 350 p.

The generating function for the higher order Airy point processes and a vector-valued Painlevé II hierarchy

Mattia Cafasso and Sofia Tarricone, LAREMA, Universite d'Angers / Concordia University, Montreal

In this talk we are going to prove that the Fredholm determinants associated to the higher order Airy kernels with several discontinuities admits a Tracy-Widom formula in terms of particular solutions of a vector-valued Painlevé II hierarchy. These Fredholm determinants coincide with the generating function for the Airy point processes that recently appeared in statistical mechanics models and in the study of multicritical Schur measures.

Soliton solutions of the matrix modified Korteweg-de Vries equation

Sandra Carillo, University La Sapienza, Italy / I.N.F.N. - Sez. Roma1, Italy

Soliton solutions of the matrix modified Korteweg-de Vries equation are studied. The results presented are based on recent results, joint with Cornelia Schiebold, [1]-[10], wherein operator equations are investigated. Notably, further to reveal structural properties, such as the existence of a *recursion operator* admitted by non-Abelian equations, also special solutions can be looked for.

In particular, operator equations of Korteweg-deVries type can be studied on application of Bäcklund transformations: indeed, most of the remarkable properties of the so-called *soliton equations*, such as the Korteweg-deVries (KdV) equation, are preserved under Bäcklund transformations. Indeed, results in [12, 13] remain valid when the generalisation from *scalar to operator* soliton equations is considered. That is, when the unknown in the nonlinear equation is an operator on a Banach space and, hence, the equation is non-Abelian instead than a real valued function of the two real variables x and t. The articles [3, 5] are devoted to Burgers-type operator equations while [1, 2, 4, 6, 7] are concerned about third order operator equations of KdV-type. The particular case of finite dimensional operators, which, then, admit a matrix representation is considered. Specifically, matrix solutions of the mKdV equation are constructed, via the Miura Transformation, which relates the matrix mKdV and KdV equations, in turn:

$$\boxed{V_t \;=\; V_{xxx} - 3\{V^2, V_x\}} \overset{M}{--} \boxed{U_t \;=\; U_{xxx} + 3\{U, U_x\}}\;, \quad \text{where} \;\; M: U + V_x + V^2 = 0$$

is the Miura transformation and $\{A, B\}$ denotes the anti-commutator of the two operators A and B. Based on matrix solutions of the KdV equation, consistent with results obtained by Goncharenko [15], matrix mKdV solutions, according to [2, 10], are constructed. In particular, examples of 2×2 and 3×3 [8, 9] matrix solutions are presented. Finally, work in progress and perspectives are listed.

References

[1] S. Carillo and C. Schiebold, Non-commutative KdV and mKdV hierarchies via recursion methods, J. Math. Phys. 50 (2009), 073510. doi: 10.1063/1.3155080

[2] S. Carillo and C. Schiebold, Matrix Korteweg-de Vries and modified Korteweg-de Vries hierarchies: Non-commutative soliton solutions, J. Math. Phys. 52 (2011), 053507. doi: 10.1063/1.3576185 27-37

[3] S. Carillo, C. Schiebold, On the recursion operator for the noncommutative Burgers hierarchy, J. Nonlinear Math. Phys., 19 n.ro 1, pp. 27-37 (2012); doi: 10.1142/S1402925112500039

[4] S. Carillo, M. Lo Schiavo and C. Schiebold, Bäcklund Transformations and Non Abelian Nonlinear Evolution Equations: a novel Bäcklund chart, SIGMA, 12 (2016), 087, 17 pages,doi: 10.3842/SIGMA.2016.087.

[5] S. Carillo, M. Lo Schiavo and C. Schiebold, Recursion Operators admitted by non-Abelian Burgers equations: Some Remarks, Mathematics and Computers in Simulation, **147C**, (2018) pp. 40–51. doi: 10.1016/j.matcom.2017.02.001

[6] S. Carillo, M. Lo Schiavo, E. Porten, C. Schiebold, A novel noncommutative KdV-type equation, its recursion operator, and solitons, J. Math. Phys., **59**, (4), (2018), 14 pp. doi: 10.1063/1.5027481.

[7] S. Carillo, M. Lo Schiavo, C. Schiebold, Abelian versus non-Abelian Bäcklund Charts: some remarks, Evolution Equations and Control Theory, 8 (1), (2019), 43-55. doi: 10.3934/eect.2019003.

[8] S. Carillo, M. Lo Schiavo, C. Schiebold, Matrix solitons solutions of the modified Korteweg-de Vries equation, NODYCON 2019 Springer Proceedings, B. Balachandran, J. Ma, W. Lacarbonara, G.Quaranta, J. Machado, G. Stepan, Ed.s, (2020), 75-83. doi: 10.1007/978-3-030-34713-0_8.

[9] S. Carillo, C. Schiebold, Construction of soliton solutions of the matrix modified Korteweg-de Vries equation, NODYCON 2021 Springer Proceedings, (NODYCON 2021), W. Lacarbonara, et al. Ed.s, in press, (2021), arXiv:2011.12677.

[10] S. Carillo, M. Lo Schiavo, C. Schiebold, N-soliton matrix mKdV solutions: a step towards their classification, preprint, 2021.

[11] S. Carillo, KdV-type equations linked via Bäcklund transformations: remarks and perspectives, Applied Numerical Mathematics, 141 (2019) pp. 81-90. doi: 10.1016/j.apnum.2018.10.002.

[12] A.S. Fokas, B. Fuchssteiner, Bäcklund transformation for hereditary symmetries, Nonlin. Anal., Theory Methods Appl. 5, 4 (1981), 423-432. doi: 10.1016/0362-546X(81)90025-0.

[13] B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations, Nonlin. Anal., Th. Meth. Appl. 3, 6 (1979), 849-862. doi: 10.1016/0362-546X(79)90052-X.
[14] B. Fuchssteiner and S. Carillo, Soliton structure versus singularity analysis: Third order completely integrable nonlinear equations in 1+1-dimensions, Phys. A 154 (1989), 467-510. doi: 10.1016/0378-4371(89)90260-4.
[15] V. M. Goncharenko, Multisoliton solutions of the matrix KdV equation, Theoret. Math. Phys., 126,81-91 (2001). doi: 10.1023/A:1005254131618.

The Gevrey type in germs of analytic functions for formal solutions of higher order holomorphic PDEs

Sergio Alejandro Carrillo Torres, Universidad Sergio Arboleda, Colombia

The aim of the talk is to study conditions for existence and uniqueness for formal power series solutions of systems of PDEs of the form

$$P(x)^k L_k(y)(x) + \cdots + P(x)L_1(y)(x) = F(x, y),$$

where $x \in (\mathbb{C}^d, 0)$, each L_j is a linear differential operator in x of order j, all coefficients are holomorphic in x and $y \in \mathbb{C}^N$, and $P(0) = 0$. The key point to solve this problem lies on the divisibility by P of certain functions $L_j^*(P)$ that naturally appear when searching for formal solutions written as power series in the variable $t = P$. Moreover, this conditions will determine the P-Gevrey class of the solution. We also include the case of analytic solutions, under a suitable Poincaré condition. The current work extendes previous results for the case $k = 1$. This is a joint work with A. Lastra (Universidad de Alcalá, Spain).

Asymptotic behavior of special function solutions of Painlevé II

Alfredo Deaño, Universidad Carlos III de Madrid, Spain

We present asymptotic expansions for special function solutions (SFS) of the Painlevé II differential equation. These SFS can be constructed as Wronskian determinants with a seed function that is a combination of Airy functions. Using the integral representations for these Airy functions and the classical method of steepest descent, we compute asymptotic expansions for SFS of Painlevé II, as the variable z tends to infinity in different sectors of the complex plane.

Strong Asymptotic of Orthogonal Polynomials with varying measures

Luis Giraldo González Ricardo, Universidad Carlos III de Madrid, Spain

In this talk we present a recent result on the strong asymptotic in L_2 sense of orthogonal polynomials with varying measures on an interval of the real line and apply it to

Cauchy biorthogonal polynomials. This is a joint work with G. López Lagomasino.
Keywords: biorthogonal polynomials, Hermite-Padé approximation, varying measures, strong asymptotic

References

[1] González Ricardo, L.G; López Lagomasino, G.: Strong asymptotic of Cauchy biorthogonal polynomials and orthogonal polynomials with varying measures. arXiv: 2102.03678.
[2] H. Widom. Extremal Polynomials Associated with a System of Curves in the Complex Plane. Advances in Mathematics **3** (1969), 127-232.

On structured matrices associated to Sobolev-type orthogonal polynomials and canonical perturbations of measures

Carlos Hermoso, University of Alcalá, Spain

It has been described in the literature that the so called "*Sobolev-type*" orthogonal polynomials are associated to $(2N+1)$-banded symmetric semi-infinite matrices corresponding to higher-order recurrence relations satisfied by them. In this talk we link these $(2N+1)$-banded matrices with the Jacobi matrices associated with the three-term recurrence relation satisfied by the standard 2-iterated Christoffel sequence of orthonormal polynomials with respect to the measure. We also stablish several connection formulas using the well known properties as basis of the orthogonal polynomial families involved. This is a joint work with E. J. Huertas, A. Lastra and F. Marcellán.

Exact WKB analysis for a holonomic system satisfied by an oscillatory integral

Sampei Hirose, Shibaura Institute of Technology, Japan

In this talk, we consider the holonomic system

$$\mathcal{A}_n : \begin{cases} \left\{(n+1)\eta^{-n}\dfrac{\partial^n}{\partial x_1^n} + (n-1)x_{n-1}\eta^{-n+2}\dfrac{\partial^{n-2}}{\partial x_1^{n-2}} + \cdots + x_1\right\}\psi = 0 \\ \left(\eta^{-1}\dfrac{\partial}{\partial x_2} - \eta^{-2}\dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0 \\ \qquad\vdots \\ \left(\eta^{-1}\dfrac{\partial}{\partial x_{n-1}} - \eta^{-n+1}\dfrac{\partial^{n-1}}{\partial x_1^{n-1}}\right)\psi = 0 \\ \left(\eta^{-1}\dfrac{\partial}{\partial x_0} - 1\right)\psi = 0 \end{cases}$$

which the oscillatory integral

$$\psi = \int e^{\eta f(t,x)}dt, \quad f(x,t) = t^{n+1} + x_{n-1}t^{n-1} + \cdots + x_1 t + x_0$$

satisfies. The purpose of this talk is to discuss the relationship between the Stokes geometry for the system $\mathcal{A}_n$ and the steepest descent path of the integral ψ, and the Borel summability of WKB solutions of the system $\mathcal{A}_n$.

Equality of ultradifferentiable classes by means of indices of mixed O-regular variation

***Javier Jiménez-Garrido* ,** Universidad de Cantabria, Spain

In the theory of ultradifferentiable function spaces there exist two classical approaches in order to control the growth of the derivatives of the functions belonging to such classes: either one uses a weight sequence or a weight function. Motivated by the comparison of both methods, that in general are mutually distinct, ultradifferentiable classes defined in terms of weight matrices have been introduced. In this general framework, one is able to treat both classical settings in a uniform and convenient way but also more classes. In this talk, we characterize the equality between ultradifferentiable function classes defined in terms of abstractly given weight matrices and in terms of the corresponding matrix of associated weight functions by using new growth indices. These indices, defined by means of weight sequences and (associated) weight functions, are extending the notion of O-regular variation to a mixed setting. Hence we are extending the known comparison results concerning classes defined in terms of a single weight sequence and of a single weight function and give also these statements an interpretation expressed in O-regular variation.

Joint work with Javier Sanz (Universidad de Valladolid, Spain) and Gerhard Schindl (Universitat of Wien, Austria).

Mehler-Heine asymptotics for some q-hypergeometric polynomials

Juan F. Mañas–Mañas, Universidad de Almería, Spain

The basic q-hypergeometric function ${}_r\phi_s$ is defined by the series

$$ {}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}, \qquad (1)$$

where $0<q<1$ and $(a_j;q)_k$ and $(b_j;q)_k$ denote the q-analogues of the Pochhammer symbol.

When one of the parameters a_j in (1) is equal to q^{-n} the basic q-hypergeo-metric function is a polynomial of degree at most n in the variable z. Our objective is to obtain a type of local asymptotics, known as Mehler–Heine asymptotics, for q-hypergeometric polynomials when $r=s$.

Concretely, by scaling adequately these polynomials we intend to get a limit relation between them and a q-analogue of the Bessel function of the first kind. Originally, this type of local asymptotics was introduced for Legendre orthogonal polynomials (OP) by the German mathematicians H. E. Heine and G. F. Mehler in the 19th century. Later, it was extended to the families of classical OP (Jacobi, Laguerre, Hermite), and more recently, these formulae were obtained for other families as discrete OP, generalized Freud OP, multiple OP or Sobolev OP, among others.

These formulae have a nice consequence about the scaled zeros of the polynomials, i.e. using the well–known Hurwitz's theorem we can establish a limit relation between these scaled zeros and the ones of a Bessel function of the first kind. In this way, we are looking for a similar result in the context of the q-analysis and we will illustrate the results with numerical examples.

This is a joint work with Juan J. Moreno–Balcázar.

Summable solutions of the Goursat problem for some inhomogeneous PDEs

Sławomir Michalik, Cardinal Stefan Wyszynski University, Poland

In this talk we consider the Goursat problem for inhomogeneous general linear partial differential equations with constant coefficients in two complex variables (t, z) in Gevrey spaces $\mathcal{O}[[t]]_s$ for some $s \geq 0$

$$\begin{cases} P(\partial_t, \partial_z)u(t,z) = f(t,z) \in \mathcal{O}_s[[t]], \\ u(t,z) - v(t,z) = O(t^j z^\alpha), \ \ v(t,z) \in \mathcal{O}_s[[t]]. \end{cases} \quad (1)$$

We show the result about analytic continuation of the solutions of (1) in the case when $s = 0$ and the Newton polygon $N(P)$ of the operator $P(\partial_t, \partial_z)$ has exactly one horizontal and one vertical side.

We also find the conditions for k-summable solutions of (1) in terms of the analytic continuation property of the Borel transform of the inhomogeneity $f(t, z)$ and the Goursat data $v(t, z)$ under assumption that the Newton polygon $N(P)$ has exactly one side with a positive slope, which is equal to $k = 1/s > 0$.

The presented results are based on [1].

References

[1] S. Michalik, *Summable solutions of the Goursat problem for some partial differential equations with constant coefficients*, submitted 22 pp., arXiv:2103.15879.

Special solutions to the multiplicative type discrete KdV equation

Nobutaka Nakazono, Institute of Engineering, Tokyo University of Agriculture and Technology, Japan

The Korteweg-de Vries (KdV) equation [1]:

$$u_t + 6uu_x + u_{xxx} = 0,$$

where $u = u(t,x) \in \mathbb{C}$ and $(t,x) \in \mathbb{C}^2$, is known as a mathematical model of waves on shallow water surfaces. The KdV equation is an important equation that has been studied extensively in physics, engineering and mathematics, especially in the field of integrable systems. In 1977, Hirota found the following integrable discrete version of the KdV (dKdV) equation [2]:

$$u_{l+1,m+1} - u_{l,m} = \frac{1}{u_{l,m+1}} - \frac{1}{u_{l+1,m}},$$

where $u_{l,m} \in \mathbb{C}$ and $(l,m) \in \mathbb{Z}^2$.

In this talk, we focus on two multiplicative-discrete versions of the dKdV equation. One is

$$u_{l+1,m+1} - u_{l,m} = \frac{\beta_{m+1} - \alpha_l}{u_{l,m+1}} - \frac{\beta_m - \alpha_{l+1}}{u_{l+1,m}}, \tag{1}$$

and the other is

$$u_{l+1,m+1} - u_{l,m} = \frac{B_{m+1} - A_l}{u_{l,m+1}} - \frac{B_m - A_{l+1}}{u_{l+1,m}}, \tag{2}$$

where

$$A_l = \frac{(1-\alpha_l)(\gamma - \alpha_l)}{\alpha_l}, \quad B_m = \frac{(1-\beta_m)(1-\gamma\beta_m)}{\beta_m}.$$

Here,

$$\alpha_l = \epsilon^l \alpha_0, \quad \beta_m = \epsilon^m \beta_0,$$

and $\alpha_0, \beta_0, \gamma \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^*$ are parameters. We show that Equations (1) and (2) have special solutions expressed in terms of solutions of the following discrete Painlevé equations:

$$q\text{-P}_{\rm III}^{D_7^{(1)}} : \overline{G}G = \frac{1 + t^{-1}F}{F(1 + {c_1}^{-1}F)}, \quad \overline{F}F = \frac{c_1(1+\overline{G})}{\overline{G}^2}$$

$$q\text{-P}_{\rm III} : \overline{G}G = \frac{c_1(1+tF)}{F(t+F)}, \quad \overline{F}F = \frac{c_1(1+c_2t\overline{G})}{\overline{G}(c_2t+\overline{G})}$$

$$q\text{-P}_{\rm V} : \begin{cases} \overline{G}G = \dfrac{(c_1+tF)(c_2+tF)}{{c_3}^2t^2(c_3+F)}, \\[2ex] \overline{F}F = \dfrac{{c_3}^2({c_3}^{-1}+qt\overline{G})(c_1c_2{c_3}^{-2}+t\overline{G})}{qt^2({c_3}^{-1}+\overline{G})} \end{cases}$$

$$
q\text{-P}_{\text{VI}} \ : \begin{cases} \overline{G}G = \dfrac{(F + c_1 t^{-4})(F + {c_1}^{-1} t^{-4})}{(F + c_2)(F + {c_2}^{-1})}, \\ \overline{F}F = \dfrac{(\overline{G} + q^{-2} c_3 t^{-4})(\overline{G} + q^{-2} {c_3}^{-1} t^{-4})}{(\overline{G} + c_4)(\overline{G} + {c_4}^{-1})} \end{cases}
$$

Note that $t \in \mathbb{C}^*$ plays the role of an independent variable, $F(t), G(t) \in \mathbb{C}$ play the roles of dependent variables and $c_i, q \in \mathbb{C}^*$ play the roles of parameters. Moreover, we here adopt the following shorthand notations for the dependent variables:

$$
F = F(t), \quad G = G(t), \quad \overline{F} = F(qt), \quad \overline{G} = G(qt).
$$

This talk is based on [3].

References

[1] D. J. Korteweg and G. De Vries. Xli. on the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 39(240):422–443, 1895.

[2] R. Hirota. Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation. J. Phys. Soc. Japan, 43(4):1424–1433, 1977.

[3] N. Nakazono. Discrete Painlevé transcendent solutions to the multiplicative type discrete KdV equations. arXiv:2104.11433 [nlin.SI].

Differential transcendence of solutions for q-difference equation of Ramanujan function

Hiroshi Ogawara, Kumamoto University, Kumamoto, Japan

In this talk, we consider the following linear q-difference equation,

$$
qx\tau^2 y - \tau y + y = 0, \tag{1}
$$

where $q \in \mathbb{C}\backslash\{0\}$ is not a root of unity and τ the q-shift operator, $\tau : \varphi(x) \mapsto \varphi(qx)$. The equation (1) has a fundamental system of solutions around $x = 0$,

$$
\begin{aligned}
y_1(x) &= {}_0\varphi_1(-; 0; q, -qx), \\
y_2(x) &= \theta_q(x) {}_2\varphi_0(0, 0; -; q, -\frac{x}{q}).
\end{aligned}
$$

The solution $y_1(x)$ is called the Ramanujan function and denoted by $A_q(x)$, which appears in Ramanujan's lost notebook [2]. The Ramanujan function $A_q(x)$ is written in the form

$$
A_q(x) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} (-x)^n.
$$

Note that $A_q(-1) = \sum_{n=0}^{\infty} q^{n^2}/(q;q)_n$ and $A_q(-q) = \sum_{n=0}^{\infty} q^{n^2+n}/(q;q)_n$ appear in the Rogers-Ramanujan identities.

We show that any non-trivial solution for (1) satisfies no non-trivial algebraic differential equation over the rational function field $\mathbb{C}(x)$. Our proof is based on the result of Nishioka [1], which is a criterion for differential transcendence of solutions for difference Riccati equations.

References

[1] Nishioka, S. Differential transcendence of solutions of difference Riccati equations and Tietze's treatment. J. Algebra 511 (2018), 16–40.

[2] Ramanujan, S. The lost notebook and other unpublished papers. Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988. With an introduction by George E. Andrews.

Towards analysis of rigid Pfaffian systems

Toshio Oshima, Center for Mathematics and Data Science, Josai University, Tokyo, Japan

We study the rigid system

$$\frac{du}{dx} = A(x)u \qquad (A(x) \in M(N, \mathbb{C}(x))$$

without a ramified irregular singular point. It will be explained following to the contents:

§1 Rigid equations and middle convolution
– Regular singularities
– Irregular singularities
– Single equations and Pfaffian systems
§2 Pfaffian system of Gauss hypergeometric equation
– Global equivalence with respect to parameters
§3 Asymptotic behavior of solutions
– Riemann-Liouville integral
§4 Middle convolution of versal unfolding
– Existence of versal unfolding
§5 Versal unfolding of systems with several variables
– Example

References

[1] T. Oshima, *Fractional calculus of Weyl algebra and Fuchsian differential equations*, MSJ Memoirs **28** (2012), Math. Soc. Japan.

[2] T. Oshima, *Transformation of KZ type equations*, Microlocal Analysis and Singular Perterbation Theory, RIMS Kokyuroku Bessatsu **B61** (2017), 141-162.

[3] T. Oshima, *Versal unfolding of irregular singurarities of a linear differential equation on the Riemann sphere*, to appear in Publ. RIMS Kyoto Uinv.

Extremal polynomials with respect to Sobolev norms

Javier Quintero Roba, Universidad Carlos III de Madrid, Spain

The first part of this talk complements previous results on characterization of polynomials of least deviation from zero in Sobolev p-norm $(1 < p < \infty)$ for the case $p = 1$. Some relevant examples are indicated. The second part deals with the location of zeros of polynomials of least deviation in discrete Sobolev p-norm. The asymptotic distribution of zeros is established on general conditions.

Joint work with Héctor Pijeira Cabrera and Abel Díaz González (Universidad Carlos III de Madrid).

Homogenization approach applied to micropolar elastic heterogeneous media

Reinaldo Rodríguez-Ramos and Victor Yanes, Universidad de La Habana, Cuba

This work deals with the applying of the two scale asymptotic homogenization method (AHM) to an inhomogeneous body with periodic structure. The homogenization process is understood as a method for representing the solution of the problem in terms of an asymptotic expansion, which allow to convert the original rapid oscillating coefficients involved in the original equations to an equivalent problem for a body with homogeneous properties, making an asymptotic analysis and seeking an averaged formulation. In other words, homogenization extracts constant parameters from very fast oscillating coefficients defined in an heterogeneous media. The method is based on the consideration of two length scales associated with the microscopic and macroscopic phenomena. The theory of micropolar elasticity is a generalization of the classic theory of elasticity that incorporates three degrees of freedom (DoF) to describe the local reorientation of the microstructure in addition to three DoF to describe the displacement at the macroscale. In the work we develop a step-by-step scheme of the AHM for micropolar 3D materials, tarting from the statement of the problem and based on microscopic-macroscopic description, the corresponding local problems, the homogenized problem and the effective coefficients as functions on the local functions are obtained (in that order).

Micropolar bi-laminated composites which satisfied the centro-symmetric postulate are studied, specifically, the case of a layer with isotropic constituents. The expressions of the effective properties for such symmetries are reported and numerical values of the stiffness, torque, Poisson's ratio, Young's modulus, twist Poison's ratio and torsional modulus are computed and presented for different volume fractions of the constituents.

Joint work with Federico J. Sabina (Universidad Nacional Autónoma de México), Yoanh Espinosa-Almeyda (Universidad Nacional Autónoma de México), Jose A. Otero (Tecnológico de Monterrey).

An approach toward Borel summability of WKB solutions of higher-order linear ODEs

Shinji Sasaki, Shibaura Institute of Technology, Japan

Since the discovery of new Stokes curves by Berk-Nevins-Roberts, WKB analysis of higher-order linear ODEs are known to be extremely difficult. The most fundamental problem, namely Borel summability of WKB solutions, is not yet solved in general.

In this talk, the speaker gives an approach toward Borel summability of WKB solutions, taking the equation of Berk-Nevins-Roberts as an example.

References

[1] T. Aoki. Toword exact WKB analysis of holonomic systems (in Japanese). RIMS Kôkyûroku, 1433:1–8, 2005.
[2] H. Berk, W. M. Nevins, and K. Roberts. New Stokes' line in WKB theory. Journal of Mathematical Physics, 23(6):988–1002, 1982.
[3] S. Hirose. Exact WKB analysis of the holonomic system satisfied by the Pearcey integral (in Japanese). Master's thesis, Kyoto University, 2010.

Unfolding of the monodromy at a resonant singularity

Tsvetana Stoyanova, Sofia University, Bulgaria

With this talk we begin a study on the unfolding of the analytic invariants of a special second-order reducible linear ordinary differential equation. In the present talk we consider the unfolding of the monodromy at a resonant singularity at the origin.

Summability of formal solutions for a family of linear moment integro-differential equations

Maria Suwińska, Cardinal Stefan Wyszyński University, Warsaw, Poland

The main point of interest during this talk will be a linear moment integro-differential equation of the form:

$$\left(1-\sum_{i\in\mathcal{K}}\sum_{q=0}^{p_i}a_{iq}(z)\partial_{m_1,t}^{-i}\partial_{m_2,z}^{q}\right)u(t,z)=\hat{f}(t,z), \tag{1}$$

with $\mathcal{K}\subset\{1,2,\ldots,\kappa\}$ for a certain $\kappa\geq 1$, all $a_{ij}(z)$ holomorphic in a certain neighborhood of the origin and $\partial_{m_1,t}$, $\partial_{m_2,z}$ denoting moment differential operators. We will use usual tools of the summability theory like Laplace and Borel transforms and the Newton polygon to show – under certain additional conditions – a connection between the summability of the formal solution $\hat{u}(t,z)$ of (1) with respect to t and the summability of $\hat{f}(t,z)$ and $\partial^q_{m_2,z}\hat{u}(t,z)$ for $0 \le q < p_\kappa$. It is a direct generalization of existing results for standard PDEs with variable coefficients.

The talk is based on [2], where strongly regular sequences are utilized in place of standard Gevrey sequences $n!^s$. Similar results for a less complex problem, also in the framework of strongly regular sequences, have been presented in [1].

Joint work with A. Lastra and S. Michalik.

References
[1] A. Lastra, S. Michalik, M. Suwińska, Summability of formal solutions for some generalized moment partial differential equations, Results Math. 76 (2021).
[2] A. Lastra, S. Michalik, M. Suwińska, Summability of formal solutions for a family of generalized moment integro-differential equations, 2020, submitted (arXiv:2012.12323).

Uniqueness of the solution of some nonlinear singular partial differential equations

Hidetoshi Tahara, Sophia University, Tokyo, Japan

In this talk, I will consider the following nonlinear singular partial differential equation

$$\text{(E)} \qquad (t\partial_t)^m u = F(t,x,\{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\le m, j<m})$$

in the complex domain. In the paper [Tahara, J. Math. Soc. Japan, 72 (2020)], I have shown some uniqueness results of the solution in the case $m=1$, by using a method similar to Cauchy's method of characteristics (which is peculiar to the first order equations). The purpose of this talk is to show that this method can be extended to the case $m=2$and we have a uniqueness of the solution of (E) in the case $m=2$ under the condition

$$\varlimsup_{R\to+0}\left[\lim_{r\to+0}\left(\frac{1}{R^4}\sup_{S\times D}|u(t,x)|\right)\right] = 0,$$

where S is a sector with finite radius at $t=0$ and D ia a neighborhood of $x=0$. In particular, we have the uniqueness of the solution under

$$\sup_{x\in D}|u(t,x)| \longrightarrow 0 \quad (\text{as } t \longrightarrow 0).$$

In my opinion, it is impossible to apply our method to the case $m \ge 3$.

On q-Painlevé equation and q-Heun equation

Kouichi Takemura, Ochanomizu University, Japan

Heun's differential equation is the second order linear differential equation give as

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-t)}y = 0,$$

with the condition $\gamma+\delta+\epsilon = \alpha+\beta+1$. This is a standard form of the Fuchsian differential equation with four singularities $\{0, 1, t, \infty\}$. A q-analogue of Heun's differential equation was given by Hahn (1971) as

$$\{a_2 z^2 + a_1 z + a_0\}y(z/q) - \{b_2 z^2 + b_1 z + b_0\}y(z) + \{c_2 z^2 + c_1 z + c_0\}y(qz) = 0,$$

with the condition $a_2 a_0 c_2 c_0 \neq 0$, which we call the q-Heun equation.

In the talk, we discuss a relationship between the q-Heun equation and the q-Painlevé VI equation. Jimbo and Sakai (1996) introduced a q-analogue of the sixth Painlevé differential equation, which is called the q-Painlevé VI or q-$P(D_5^{(1)})$. They obtained q-Painlevé VI by connection preserving deformation of a certain linear q-difference equation, which is equivalent to

$$\begin{aligned} &\left\{\frac{z(g\nu_1-1)(g\nu_2-1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{fg}\right\}y(z) \\ &+\frac{\nu_1\nu_2(z-q\nu_3)(z-q\nu_4)}{q(qf-z)}\{gy(z)-y(z/q)\} \\ &+\frac{(z-\kappa_1/\nu_7)(z-\kappa_1/\nu_8)}{q(f-z)}\left\{\frac{y(z)}{g}-y(qz)\right\}=0, \end{aligned} \tag{1}$$

with the constraint $\kappa_1^2\kappa_2^2 = q\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8$. This is different from the q-Heun equation. On the other hand, the space of initial conditions is known for each q-Painlevé equation. We can construct the space of initial conditions for q-$P(D_5^{(1)})$ by applying blowing-up at the eight specific points. In this talk, we also apply the calculation of the blowing-up for the equation (1). Then we obtain the q-Heun equation.

As another topic, we explain a q-integral transformation of solutions to the q-Heun equation, which is related to the q-middle convolution introduced by Sakai and Yamaguchi. This is a q-analogue of the Euler integral transformation of solutions to Heun's differential equation,

This talk is mainly based on a joint work with Shoko Sasaki and Shun Takagi.

Parametric Stokes phenomena of the Gauss hypergeometric differential equation with a large parameter

Mika Tanda, Kwansei University, Japan

Stokes phenomena with respect to parameters are investigated for the Gauss hypergeometric differential equation with a large parameter η:

$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q\Big)\psi=0, \tag{1}$$

where $Q=Q_0+\eta^{-1}Q_1+\eta^{-2}Q_2$,

$$\begin{aligned}
Q_0&=\frac{(\alpha-\beta)^2x^2+2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2},\\
Q_1&=\frac{(\alpha-\beta)(\alpha_0-\beta_0)x^2+(2(\alpha_0\beta+\alpha\beta_0)-\beta\gamma_0-\beta_0\gamma-\alpha\gamma_0-\alpha_0\gamma+\gamma)x+\gamma(\gamma_0-1)}{2x^2(x-1)^2},\\
Q_2&=\frac{(\alpha_0-\beta_0+1)(\alpha_0-\beta_0-1)x^2+2(2\alpha_0\beta_0-\beta_0\gamma_0-\alpha_0\gamma_0+\gamma_0)x+\gamma_0(\gamma_0-2)}{4x^2(x-1)^2}.
\end{aligned}$$

The Parametric Stokes phenomena mean Stokes phenomena associated with a change of parameters contained in the equation. The equation (1) has formal solutions which are called the WKB solutions. In this talk, the parametric Stokes phenomena of (1) is described in terms of the WKB solutions. The WKB solutions are Borel summable in a region surrounded by Stokes curves if the Stokes geometry is non-degenerate. If it is the case, we can obtain analytic solutions by taking the Borel sums of the WKB solutions. These analytic solutions can be analytically continued with respect to the parameters. To get the Stokes phenomena, the notion of the Voros coefficient is introduced for (1). The explicit forms of the Voros coefficients are given as well as their Borel sums. By using them, formulas which describe the Stokes phenomena are obtained.

This is a collaboration with Takashi Aoki and Toshinori Tanahashi.

Voros coefficients at the origin and at the infinity of the generalized hypergeometric differential equation with a large parameter

Shofu Uchida, Graduate School of Science and Engineering, Kindai University, Japan

The notion of the Voros coefficients is one of the keys in the exact WKB analysis of differential equations with a large parameter. It has been introduced mainly for second-order ordinary differential equations and used effectively in the descriptions of the parametric Stokes phenomena and of the relations between Borel resummed WKB solutions and classical special functions. The Voros coefficients of the origin and at the infinity are defined and their explicit forms are given for the generalized hypergeometric differential equation with a large parameter.

References

[1] T. Aoki, S. Uchida: Voros coefficients at the origin and at the infinity of the generalized hypergeometric differential equations with a large parameter: arXiv:2104.13751 [math.CA].

Multiple movable singularity of some Hamiltonian system -Application of Borel summability-

Masafumi Yoshino, Hiroshima University, Japan

We study the construction of solution with multiple movable singular points of some Hamiltonian system including Painlevé I, II, IV equations. By transforming our equation to the integrable equation we give a new proof of the existence of a solution containing multiple singular points. Indeed, in the Painlevé case we transform the Hamiltonian to that of the elliptic function. The transformation is constructed by virtue of the (parametric) global Borel summability of a certain first order partial differential equation.

On the powers of the power series $\sum_{n\geq 0} q^{n(n-1)/2}(-x)^n$ where $|q| > 1$

Changgui Zhang, University of Lille, France

The main subject of this talk is to explain how to obtain a linear q-difference equation satisfied by the m-th power $\left(\sum_{n\geq 0} q^{n(n-1)/2}(-x)^n\right)^m$ where m is an integer ≥ 1. For doing that, we will start with some general results on the analytic theory of q-difference equations.

The trigonometric Airy functions (homogeneous and inhomogeneous)

Federico Zullo, Università di Brescia, Italy

In this talk I will illustrate the properties of some solutions of the homogeneous and inhomogeneous Airy equations. These solutions present different properties generalizing those of the trigonometric functions and the analogies will be analyzed by different points of view. Also, a representation through integers will be given.

List of Participants

Shunya Adachi	(Kumamoto University, Japan)
Dennis Bacani	(Temple University, Japan)
Alexander Batkhin	(Keldysh Institute of Applied Mathematics of RAS, Russia)
Bjorn Berntson	(KTH Royal Institute of Technology, Sweden)
Alexander Bruno	(Keldysh Institute of Applied Mathematics of RAS, Russia)
Mattia Cafasso	(Université d'Angers, France)
Sandra Carillo	(University La Sapienza, Italy)
Sergio Alejandro Carrillo Torres	(Universidad Sergio Arboleda, Colombia)
Ewa Ciechanowicz	(University of Szczecin, Poland)
Peter Clarkson	(University of Kent, United Kingdom)
Anibal Coronel	(Universidad del Bio-Bio, Chile)
Alfredo Deaño	(Universidad Carlos III de Madrid, Spain)
Thomas Dreyfus	(CNRS, Université de Strasbourg, France)
Galina Filipuk	(University of Warsaw, Poland)
Luis Giraldo González Ricardo	(Universidad Carlos III de Madrid, Spain)
Renat Gontsov	(Institute for Information Transmission Problems of RAS, Russia)
Irina Goryuchkina	(Keldysh Institute of Applied Mathematics, RAS, Russia)
Davide Guzzetti	(SISSA, Italy)
Carlos Hermoso	(University of Alcalá, Spain)
Sampei Hirose	(Shibaura Institute of Technology, Japan)
Luan Hoanf	(Texas Tech University, USA)
Edmundo J. Huertas	(University of Alcalá, Spain)
Hideaki Izumi	(Chiba Institute of Technology, Japan)
Javier Jiménez Garrido	(Universidad de Cantabria, Spain)
Thomas Kecker	(University of Portsmouth, United Kingdom)
Hironobu Kimura	(Kumamoto University, Japan)
Martin Klimes	(FER, University of Zagreb, Croatia)
Alberto Lastra	(University of Alcalá, Spain)
Jose Ernie Lope	(University of the Philippines Diliman, Philippines)
Stephane Malek	(University of Lille, France)
Juan F. Mañas-Mañas	(Universidad de Almería, Spain)

Sławomir Michalik	(Cardinal Stefan Wyszynski University, Poland)
Claude Mitschi	(Université de Strasbourg, France)
Nobutaka Nakazono	(Tokyo University of Agriculture and Technology, Japan)
Hiroshi Ogawara	(Kumamoto University, Japan)
Yasunori Okada	(Chiba University, Japan)
Toshio Oshima	(Josai University, Japan)
Mark Philip Ona	(University of the Philippines Diliman, Philippines)
Andrew Pickering	(Universidad Rey Juan Carlos, Spain)
Javier Alejandro Quintero Roba	(Universidad Carlos III de Madrid, Spain)
Reinaldo Rodríguez-Ramos	(Universidad de La Habana, Cuba)
Javier Sanz	(University of Valladolid, Spain)
Shinji Sasaki	(Shibaura Institute of Technology, Japan)
Cornelia Schiebold	(Mid Sweden University, Sweden)
Jiro Sekiguchi	(Tokyo University of Agriculture and Technology, Japan)
Patryk Szlufik	(University of Warsaw, Poland)
Tsvetana Stoyanova	(Sofia University "St. Kliment Ohridski", Bulgaria)
Maria Suwińska	(Cardinal Stefan Wyszynski University, Poland)
Hidetoshi Tahara	(Sophia University, Japan)
Kouichi Takemura	(Ochanomizu University, Japan)
Mika Tanda	(Kwansei Gakuin University, Japan)
Sofia Tarricone	(Université d'Angers, France)
Shofu Uchida	(Kindai University, Japan)
Masafumi Yoshino	(Hiroshima University, Japan)
Changgui Zhang	(University of Lille, France)
Federico Zullo	(Università di Brescia, Italy)